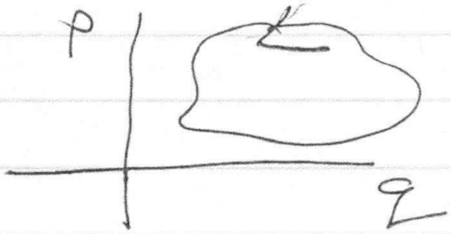
Action - Angle Variables

→ Extending the Concepts of Adiabatic Invariance.

Action-Angle Variables

(L/L → canonical variables)

Key concept: $\oint \underline{p\,dq}$



$\Downarrow$

Phase Space Circulation

$\Leftrightarrow$ Poincare - Cartan Invariant

$\lambda = \lambda(t)$
$E = E(t)$ $\Rightarrow$

$E = \text{const.}$
$\lambda = \text{const.}$
closed system

Canonical Transformations

⇒ specify transformation rules

Adiabatic Invariants

i.e. $\oint_{E,\lambda} p\,dq \to \text{COM}$ with slow parametric variation

{ circulation as const motion

Action-Angle Variables

{ circulation as variable (momentum)

⇒ integrability,

{ phase space geometry, resonance

so

Action-Angle variables $\downarrow$

⇒ seek variables (i.e. C.T.: $p, q \to \overset{P}{I}, \overset{Q}{\theta}$)

s/t:

$$H = H(I) \quad , \text{so} \quad \dot{I} = 0, \quad \dot{\theta} = \frac{\partial H}{\partial I}$$

integrable

[i.e. C.T. to conserved momentum, cyclic coordinate

$$\theta = \omega t + \theta_0$$

⇒ C.T. is equivalent to integration of system.

{ A/A are variables on which system integrated

→ crudely : integrate via new variables
s/t $I \to$ 'generalized radius'
$\theta \to$ " " angle

so

$$p, q \to \theta, I$$

$$H(p,q) \to H'(I) \qquad \dot{I} = 0 \qquad \dot{\theta} = \omega$$

C.T. : independent variables q, I
(q, P)

⇒ Type II : $F_2 = F_2(q, P)$

so $$p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}$$

⇒ $$p = \frac{\partial F_2}{\partial q},$$ [redacted] $$\theta = \frac{\partial F_2}{\partial I}$$

but $p = \frac{\partial F_2}{\partial q}$ equiv. to $p = \frac{\partial S}{\partial q}$ from H-J theory

(always, for Type II)

so can write in terms action as generating function, i.e.

$$F_2(q, P) = F_2(q, I) = S(q, I).$$

so $\Theta = \frac{\partial S_0}{\partial I}$, $p = \frac{\partial S_0}{\partial q}$

Now, further:

$S_0 = S_0(q, I)$ indep. time; i.e. $\lambda = \lambda(\cancel{t}) = \text{const}$

and [redacted] $H(q,p) \Rightarrow \underset{= E(I)}{H(I)}$, with Θ cyclic

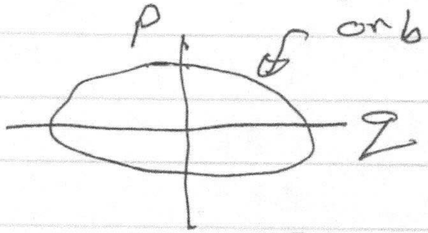
in new variables → EOM:

$$\Rightarrow \dot{I} = -\frac{\partial H}{\partial \Theta} = 0, \quad \dot{\Theta} = \frac{\partial H}{\partial I} = \underset{\substack{\downarrow \\ \text{angular} \\ \text{frequency}}}{\omega(I)}$$

$\underline{\underline{\text{i.e.}}}$ I and E constant.

Contrast: Adiabatic invariants ⇒
$I \sim$ const, E evolves as ω evolves

Now,



orbit on closed contour

$$I = \underset{\substack{\downarrow \\ \text{1 circuit} \\ \text{circulation}}}{\oint} p \frac{dq}{2\pi} = \underset{\substack{\downarrow \\ \text{phase volume}}}{\int} dp\, dq$$

another way:

$S_0 = S_0(q, I)$, $p = \partial S_0/\partial q$

$\theta = \partial S_0/\partial I$

so

$$\frac{d\theta}{dq} = \frac{\partial}{\partial q}\frac{\partial S}{\partial I} = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}$$

$$d\theta = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}\,dq$$

⇒

$$2\pi = \frac{\partial}{\partial I}\oint \frac{\partial S}{\partial q}\,dq$$

$$= \frac{\partial}{\partial I}\oint p\,dq.$$

⇒ $$\boxed{I = \oint \frac{p}{2\pi}\,dq \rightarrow \text{Action Variable}}$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial I} = \frac{\partial E(I)}{\partial I} \equiv \underset{\text{angle variable.}}{\omega(I)}}$$

I → radius

ω → winding rate, ~~frequency~~

Comparison / Contrast

Adiabatic Invariants	A-A Variables
$\lambda = \lambda(t)$, open loop	$\lambda = \lambda_0$ const, closed loop
$I = \oint_{E,\lambda} p\,dq \sim \left\{ \begin{matrix} \text{approx} \\ \text{COM} \end{matrix} \right.$	$I = \oint p\,dq \quad \left\{ \begin{matrix} \text{exact} \\ \text{COM} \end{matrix} \right.$
E varies with ω, $I \sim$ const.	E, I const. $\dot{I} = 0$ is H EOM
COM for multiple scale problems	Variables in which system is integrated i.e. $\dot{I} = 0$
1 adiabatic inv. per closed cycle (i.e. mirror) (separability implicit)	separable system ⇒ 1 action variable/ cycle.

Examples:

→ H.O.: 1D
2D
→ general 1D
→ free particle in box

1) 1D H O.

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

$$\left(\frac{\partial S}{\partial q}\right)^2 + \omega^2 q^2 = E \quad \text{is H-J.}$$

$$I = \frac{1}{2\pi}\oint \left(E - \frac{\omega^2 q^2}{2}\right)^{1/2} dq$$

$$\oint = 2\int_{q_-}^{q_+} \qquad E = \omega^2 q^2 \rightarrow \text{turning pts.}$$

$$q_\pm = \pm\sqrt{E}/\omega$$

,",

$$I = \frac{2}{2\pi}\int_{q_-}^{q_+} \left[\left(E - \frac{\omega^2 q^2}{2}\right)\right]^{1/2} dq$$

$$q = \sqrt{2E/(\omega)}\,\sin\theta \;, \quad dq = \sqrt{\frac{2E}{\omega}}\cos\theta$$

so, $I = E/\omega$

$p = I \equiv$ "new" momentum

$H = E = I\omega$ so, $\dot{\theta} = \frac{\partial H}{\partial I} = \omega$

$\theta = \omega t + \theta_0$

$S = S(q, I) =$ [redacted] $\int_{q_0}^{q} dq \left(I\omega - \frac{\omega^2 q^2}{2}\right)^{1/2}$

2) For 2D,

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2 q_1^2}{2} + \frac{\omega_2^2 q_2^2}{2}$$

$H = f(1) + f(2) = E$ separable!

$f(1) = \frac{p_1^2}{2} + \frac{\omega^2 q^2}{2} = E_1 \to$ const.

$f(2) = \frac{p_2^2}{2} + \frac{\omega^2 q^2}{2} = E_2 \to$ const.

so, for action variables I_1, I_2:

$$I_1 = \frac{1}{2\pi}\oint p_1 \, dq = \frac{1}{2\pi}\oint p_1(q_1)\, dq_1 = \frac{E_1}{\omega_1}$$

$$I_2 = E_2/\omega_2$$

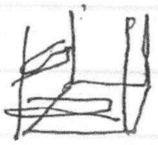
$$H(I_1, I_2) = E = E_1 + E_2$$
$$= I_1\omega_1 + I_2\omega_2$$

→ separable, so

→ additive form of H in A-A variables

2) Free Particle in 2D $\begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$ (hard wall)

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$



→ 2 Degs freedom ⇒ 2 I's, 2 θ's

$$\therefore \quad I_1 = \frac{1}{2\pi}\oint p_x \, dx$$

$$I_2 = \frac{1}{2\pi}\oint p_y \, dy$$

$$\oint p_x \, dx = \int_0^a p_{x_+} \, dx + \int_a^0 p_{x_-} \, dx$$

$p_{x_+} = -p_{x_-}$ (reverse when bounce off wall)

$$\oint p_x \, dx = 2a \, |p_x|$$

$\therefore \quad I_1 = \frac{a}{\pi} |p_x|$

$I_2 = \frac{b}{\pi} |p_y|$

so $H = E = \frac{p_x^2 + p_y^3}{2m}$

$$= \frac{\pi^2}{2m} \left(\frac{I_1^3}{a^2} + \frac{I_2^2}{b^2} \right)$$

$$\omega(I_{1,2}) = \frac{\partial E(I_1, I_2)}{\partial I_{1,2}} = \frac{\pi^2}{m} \frac{I_1}{a^2}, \frac{\pi^2}{m} \frac{I_2}{b^2}$$

2 points:

ⓐ contrast:

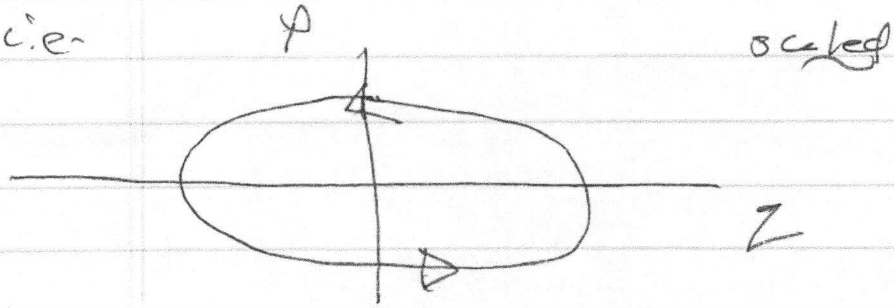
→ H.O. $\omega(I) = \omega_0 = \text{const.}$

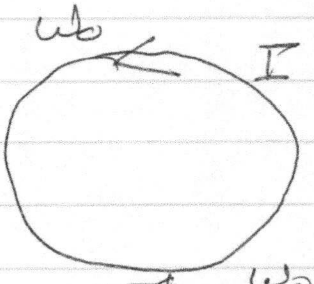
$\partial\omega/\partial I = 0$

$I\omega_0 = E$ → constant frequency

→ no shear in winding rate

i.e.





and all I contours have same rotation frequency ω_0

i.e.

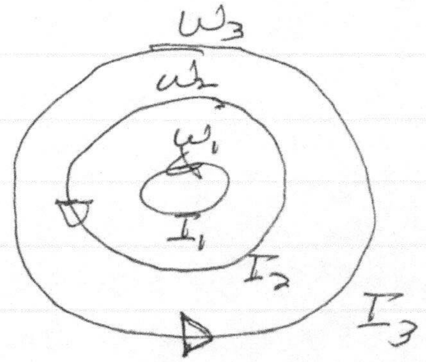
→ box $\omega(I) = \frac{\pi^2}{ma^2} I$

$\partial\omega(I)/\partial I \neq$ $\omega \sim |p|$

⇒ winding rate varies with I

⇒ "shear"

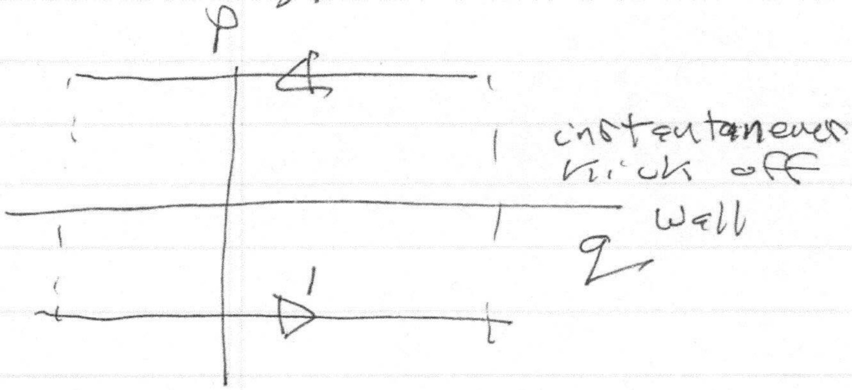
i.e

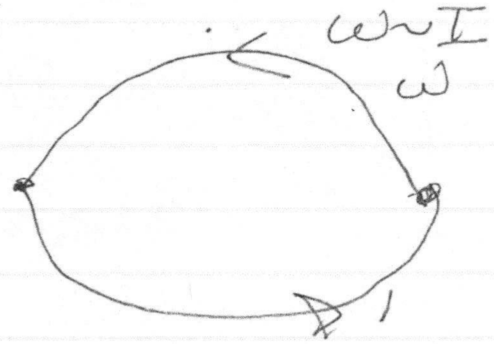


winding rate increases with I (i.e. $\omega \sim I$)

⇒ differential rotation

q, p → I, ω





i.e, top box — top circle, etc.

? H.O. is linear problem, with $\partial\omega/\partial I = 0$

Box has $\partial\omega/\partial I \neq 0$, yet is linear too ?!

why ?

n.b. Consider general 1D potential:

$$H = p^2 + V(q)$$

$$I = \oint \frac{p\,dq}{2\pi} = \frac{1}{2\pi}\oint \left[E - V(q)\right]^{1/2} dq$$

$$= \underline{I(E)}$$

$$\omega = \partial E(I)/\partial I$$

now, for $V(q) \sim \beta q^4$

$$I \sim c' E^{3/4}$$

$$\Rightarrow \quad E \sim c\, I^{4/3} \quad \text{so} \quad \omega(I) \sim c'' I^{1/3}$$

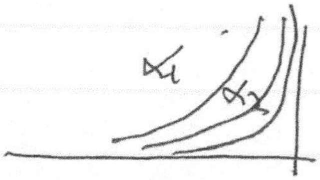
shear!

⇒ Nonlinearity develops from $V \sim \xi^{\alpha}$ potential for $\alpha > 2$.

∴ View hard wall as a limiting case

i.e.

$$V = V_0 \left(x/a\right)^{\alpha}$$



$\alpha \gg 1$

so hard wall boundary condition appears as nonlinearity due high high powers implicit in piecewise continuous potential.

② Reln. QM

classically:

$$H = E = \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right)$$

if

$$\left.\begin{array}{l} I_1 \to n\hbar \\ I_2 \to m\hbar \end{array}\right\}$$ quantize action variables

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \rightarrow$$ eigenstates of free QM particle in box

aside: Can observe correspondance

Classical	Quantum
$I = E/\omega$	$E = (N + 1/2)\,\hbar\omega$
$H = I\omega$	$\downarrow$ # quanta ⇔ occupation (quantum #)

$\therefore$ {suggests view I as classical # of excitations/waves ⇒ exciton density

straightforward to generalize: (linear wave)

$$\underbrace{I = E/\omega}_{\text{linear H.O.}} \Longrightarrow \underbrace{N\left(\underset{\underline{x}}{k,}\,\omega\right)}_{\substack{\text{Action Density} \\ \text{as Wave Density, \# waves}}} = \overset{\text{wave energy density}}{E\left(\underset{\underline{x}}{k,}\,\omega\right)} / \underbrace{\omega_k}_{\text{wave frequency}}$$

→ General Properties of Motion in s dimensions.

system

Now, consider:
- s degrees of freedom (arbitrary)
- separable H.-J. equation

$$S = \sum_{i=1}^{s} S_i(q)$$ (i.e. integrable)

∴ can define $\underline{\underline{s}}$ action variables I_i

$$I_i = \oint \frac{p_i dq_i}{(2\pi)}$$ $\underline{\text{i.e.}}$ s - IOMs.

and $\Theta_i = \partial S_0 / \partial I_i$ angle variables

$\underline{\text{so}}$

$$\dot{I}_i = 0$$

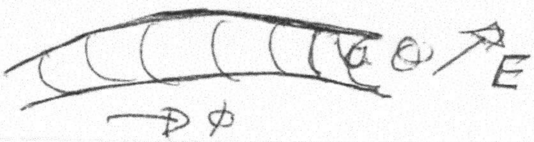
$$\dot{\Theta}_i = \omega_i(E) + +t_0$$

$$\omega_i(E) = \partial E / \partial I_i$$

$\underline{\text{i.e.}}$ for $s = 2$

$$\dot{I}_1 = \dot{I}_2 = 0 \qquad \omega_{1,2} = \partial F / \partial I_{1,2}$$

$$\Theta_{1,2} = \omega_{1,2}(E)t + t_0$$



$\left\{\begin{array}{l}\text{change } I \\ \to \text{change surf} \\ \leadsto \text{nested surfaces.}\end{array}\right.$

∴ phase space is 2 torus. Fixed $E \Rightarrow$ motion on toroidal surface.

[In general, phase space is S-torus.]

$$\theta = \omega_1(E)\,t \qquad\qquad \theta = \frac{\omega_1(E)}{\omega_2(E)}\,\phi$$
$$\phi = \omega_2(E)\,t$$

→ now, for any $F(\underline{\theta}, \underline{I})$, can write: fourier series

$$F = \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_S} A_{\ell_1, \ell_2 \ldots S} \exp\left[i(\ell_1\theta_1 + \ell_2\theta_2 + \cdots \ell_S\theta_S)\right]$$

$\ell_1, \ell_2, \ldots \ell_S$ integers. ⇒ define vector $\underline{\ell}$

equivalently:

$$F = \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_S} A_{\ell_1, \ell_2 \cdots \ell_S} \exp\left[it\left(\underline{\ell} \cdot \overset{\omega_i}{\frac{\partial E}{\partial \underline{I}}}\right)\right]$$

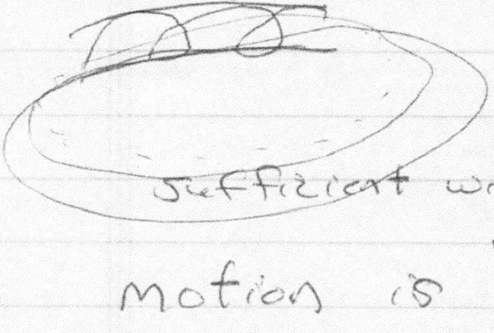
$$\underline{\ell} \cdot \frac{\partial E}{\partial I} = \ell_1 \frac{\partial E}{\partial I_1} + \ell_2 \frac{\partial E}{\partial I_2} + \cdots \ell_S \frac{\partial F}{\partial I_S}.$$

Now, in general:

→ frequencies not commensurate, so F not periodic i.e. $\underline{l} \cdot \partial E / \partial \underline{I}$ irrational

→ indeed, system generally not periodic in any coordinate (except for special E).

but, for sufficient time, system will come arbitrarily close, to starting point. → [Poincare Recurrence Thm.]



i.e. {trajectory ergodically covers surface of torus}

∴ motion is "conditionally" periodic.

But; degeneracy happens!

- degeneracy: $n\omega_i = m\omega_j$
- all ω commensurate ⇒ complete degeneracy.

So, as in Kepler problem, ⇒ degeneracy implies reduction in number of independent I_i. Why?

Commensurate frequencies ⇒

$$n_1 \omega_1 = n_2 \omega_2$$

$$n_1 \frac{\partial E}{\partial I_1} = n_2 \frac{\partial E}{\partial I_2}$$

so $E = E(n_2 I_1 + n_1 I_2)$

i.e. – energy depends on sum of action variables
linear superposition

⇒

– degeneracy

⇒

– can make canonical transformation
so $E = E(I')$, only.

⇒

∴ in degenerate motion, there is an increase in the number of one-valued integrals of the motion, relative to non-degenerate case.

i.e. non-degenerate motion – s degs freedom

$2s-1$ → IOM'S

{ s values I_i ⇒ single valued I_i
{ s−1 values a la $\theta_i \, \partial E/\partial I_k - \theta_k \, \partial E/\partial I_i$

note: $S-1$ values → phases (i.c.'s) of angle variables.

→ **not** single valued.

but if degeneracy, note though:

→ $n_1\theta_1 - n_2\theta_2$ not single valued

it **is**, to addition of 2π!

so

→ $\sin(n_1\theta_1 - n_2\theta_2)$ **is** single valued.
(etc)